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An analysis of thin toroidal shells of elliptical profile subjected to uniform pressure

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AN ANALYSIS OF THIN TOROIDAL SHELLS OF ELLIPTICAL PROFILE SUBJECTED TO UNIFORM PRESSURE

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PART 1

INTRODUCTION

In this thesis the problem of a toroidal shell of elliptical profile is considered from the point of view of the small deflection theory of thin shells of revolution loaded symmetrically with respect to their axis. Solutions of such problems are found by applying methods of asymptotic integration of a differential equation involving a large parameter multiplied by a function which vanishes at certain points.

The problem has been formulated by considering first the finite deflection theory of thin shells of revolution subjected to rotationally symmetric loads, (see Part 2). In this, all the necessary expressions for the required quantities have been shown. The notation and the method of approach of E. Reissner (6) have been adopted throughout this work. Reference (6) also contains references concerning the historical development of the subject.

Part 3 deals with the system of equations from point of view of small (linearized) deflection theory. This system of equations follows immediately from the relations derived in Part 2. Also a general expression for the resulting differential equations has been written down.

In Part 4 the basic relations developed in Part 3 are then applied to our particular problem of toroidal shell with elliptical profile. The complete calculations to derive the relations for the particular quantities have been shown in appendices. An approximate solution of the resulting differential equation has been worked out by the method of asymptotic integration suggested by R. A. Clark (2) in his recent paper.

It may be appropriate to say here that the general problem of determining rotationally-symmetric stress distribution in thin shells of revolution may be reduced to that of solving two coupled second order differential equations. Using the formulation of R. A. Clark (2) the differential equations for small deflection may be written in the form

$$X'' - \theta_1 X + \mu \Phi Y = F$$
$$Y'' - \theta_2 Y + \mu \Phi X = G$$

where prime indicates differentiation with respect to an independent variable, ξ, which is the parameter in a representation of the middle surface of shell and the coefficients θ_1, θ_2, and along with quantities F and G are functions of this parameter.

If parameter μ is large we can obtain approximate solution of the above equations by applying methods of asymptotic integration. The usual method of asymptotic integration assumes that the term involving the parameter is large everywhere. For a toroidal shell Φ vanishes at some points; consequently, the ordinary asymptotic integration theory is valid only for regions of shell where Φ is not small. For regions where Φ is small it is possible to make a separate approximation but it is much more desirable to have a single representation for a solution which is valid regardless of the size of Φ. Such a representation has been shown by R. A. Clark (2) for the solution of non-homogeneous equations. This also happens to be the case for our particular problem under consideration.

The advantage of asymptotic integration methods is that the accuracy of results so obtained increases as the parameter μ becomes large. Power series methods on the other hand become less practical as μ becomes large. However, if μ is not too large, either power series or methods of successive approximation may be used.

PART 2

FINITE DEFLECTION THEORY OF THIN SHELLS OF REVOLUTION SUBJECTED TO ROTATIONALLY SYMMETRIC LOADS

2.1 Geometry of Thin Shells of Revolution:

In a shell with rotational symmetry the middle surface is given by parametric equations

$$r = r(\xi), \qquad z = z(\xi) \tag{2.1}$$

where r, z, and angle θ are the cylindrical coordinates of middle surface shown in Figure (1). Let ϕ be the angle between r direction and the tangent to a meridian. Then,

$$r' = \alpha \cos\phi, \qquad z' = \alpha \sin\phi$$

where $$\alpha^2 = (r')^2 + (z')^2 \tag{2.2}$$

and prime indicates the differentiation with respect to parameter ξ .

Curvilinear coordinates:

Let $\bar{i}$, $\bar{j}$, $\bar{k}$ be unit vectors in x, y, and z directions respectively as shown in Figure (2). The radial and circumferential unit vectors $\bar{j}_r$ and $\bar{j}_\theta$ are then defined by

$$\bar{j}_r = \bar{i}\cos\theta + \bar{j}\sin\theta, \qquad \bar{j}_\theta = -\bar{i}\sin\theta + \bar{j}\cos\theta \tag{2.3}$$

and tangential and normal unit vectors $\bar{j}_\xi$ and $\bar{n}$ become

$$\bar{j}_\xi = \bar{j}_r\cos\phi + \bar{k}\sin\phi, \qquad \bar{n} = -\bar{j}_r\sin\phi + \bar{k}\cos\phi \tag{2.4}$$

The radius vector $\bar{R}$ to any point of the shell may now be written in the form

$$\bar{R} = r\bar{j}_r + z\bar{k} + \zeta\bar{n} \tag{2.5}$$

where ζ represents the distance of the point from the middle surface. The quantities ξ, θ, ζ define a system of orthogonal curvilinear coordinates in space. The meridional line element is of the following form (see Appendix 1):

$$ds = d\bar{R}\cdot d\bar{R}$$

$$= \alpha\left(1-\frac{\zeta\phi'}{\alpha}\right)^2 d\xi^2 + r^2\left(1-\zeta\frac{\sin\phi}{r}\right)d\theta^2 + d\zeta^2 \tag{2.6}$$

Principal radii of curvature of the middle surface:

From Figures (3) and (4) the principal radii of curvature in circumferential and meridional directions can be written as follows:

$$R_\theta = \frac{r}{\sin\phi}, \quad R_\xi = \frac{\alpha}{\phi'} \tag{2.7}$$

2.2 Equations for Strain:

If the subscript o refers to the undeformed middle surface, the parametric equations of the deformed middle surface can be written in the following form:

$$r = r_o + u, \qquad z = z_o + w \tag{2.8}$$

The quantities u and w are then the components of displacement in the radial and axial directions as shown in Figure (5). Also let

$$\phi = \phi_o - \beta \tag{2.9}$$

where β is the angle enclosed by the tangents to the deformed and undeformed meridian at the same material point. The assumption is now made that deformations due to transverse shear stress are neglected compared with the deformations due to remaining stresses.

The meridional line elements for deformed and undeformed shell are

$$ds^2 = \alpha^2\left(1-\zeta\frac{\phi'}{\alpha}\right)d\xi^2 + r^2\left(1-\zeta\frac{\sin\phi}{r}\right)^2 d\theta^2 + d\zeta^2 \tag{2.6}$$

$$ds_0^2 = \alpha_0^2\left(1-\zeta\frac{\phi_0'}{\alpha_0}\right)d\xi^2 + r_0^2\left(1-\zeta\frac{\sin\phi_0}{r_0}\right)d\theta^2 + d\zeta^2 \tag{2.10}$$

from which component of strain in meridional direction can be written as

$$e_\xi = \frac{ds_\xi - ds_{0\xi}}{ds_{0\xi}}$$

$$= \frac{(\alpha-\alpha_0) - \zeta(\phi-\phi_0)}{\alpha_0(1-\zeta/R_{\xi_0})} \tag{2.11}$$

Similarly we can write

$$e_\theta = \frac{(r-r_0) - \zeta(\sin\phi - \sin\phi_0)}{r_0(1-\zeta/R_{\theta_0})} \tag{2.12}$$

Now attention is restricted to thin shells in the sense that the thickness h is small compared with the magnitudes of the radii of curvature R_ξ and R_θ as defined by previous equations. The terms containing ζ may thus be neglected in the denominators and we get

$$e_\xi = \frac{\alpha-\alpha_0}{\alpha_0} - \frac{\zeta}{\alpha_0}(\phi-\phi_0), \quad e_\theta = \frac{r-r_0}{r_0} - \frac{\zeta}{r_0}(\sin\phi - \sin\phi_0) \tag{2.13}$$

Strain for middle surface of shell:

The strain for the middle surface in meridional direction can be written as

$$e_{\xi m} = \frac{\alpha - \alpha_0}{\alpha_0}$$

$$= \frac{r_0' + u'}{r_0'} \frac{\cos\phi_0}{\cos\phi} - 1 \tag{2.14}$$

and in circumferential direction

$$e_{\theta m} = \frac{r - r_0}{r_0} = \frac{u}{r_0} \tag{2.15}$$

The curvature change:

The curvature changes x_ξ and x_θ in meridional and circumferential directions are expressed by means of the relation of the form

$$\chi_\xi = -\frac{(\phi' - \phi_0')}{\alpha_0} = \frac{\beta'}{\alpha_0}, \quad \chi_\theta = -\frac{(\sin\phi - \sin\phi_0)}{r_0} \tag{2.16 a,b}$$

By substitution of Equations (2.14) to (2.16) into Equation (2.13) we can write the equations for strain in meridional and circumferential direction in the following form:

$$e_\xi = e_{\theta m} + \zeta \chi_\xi, \quad e_\theta = e_{\theta m} + \zeta \chi_\theta \tag{2.17}$$

Development of $e_{\xi m}$ and x_θ in terms of β:

The MacLaurin expansion for the quantities $e_{\xi m}$ and x_θ, as given in Equations (2.14) and (2.16) respectively, in terms of angle β is of the following form (see Appendix 2):

$$\chi_\theta = \beta \frac{\cos\phi_o}{r_o} + \beta^2 \frac{\sin\phi_o}{2r_o} + \ldots\ldots \tag{2.18}$$

$$e_{\xi m} = \left(\frac{u'}{r'}\right) - \beta\tan\phi_o + \beta^2\left(\frac{1}{2} + \tan^2\phi_o\right) - \left(\frac{u'\beta}{r_o'}\right)\tan\phi_o + \ldots\ldots \tag{2.19}$$

Expression for axial displacement:

The expressions for quantities e_ξ and e_θ do not contain the axial displacement component w. This is obtained from the relation

$$z' = \alpha\sin\phi \tag{2.2}$$

in the form

$$w' = \alpha\sin\phi - z_o' \tag{2.20}$$

which in view of equations $e_{\xi m} = \dfrac{\alpha - \alpha_o}{\alpha_o}$ and $\phi = \phi_o - \beta$ can be written as

$$w' = \alpha_o(1 + e_{\xi m})\sin(\phi_o - \beta) - z_o' \tag{2.21}$$

The development of w' in terms of β can be written down as (see Appendix 2)

$$w' = z_o' e_{\xi m} - r_o'(1 + e_{\xi m})\beta - \frac{z_o'}{2}(1 + e_{\xi m})\beta^2 + \ldots\ldots \tag{2.22}$$

Compatibility equations:

Finally we can write down compatibility relation from a comparison of expressions for $e_{\xi m}$ and $e_{\theta m}$ as given by Equations (2.14) and (2.15). The relevant compatibility relation is of the following form (see Appendix 2):

$$\cos\phi_o(r_o e_{\theta m})' - \cos\phi(r_o' e_{\xi m}) = r_o'(\cos\phi - \cos\phi_o) \tag{2.23}$$

and when developed in powers of becomes

$$\cos\phi_0 (r_0 \ell_{\theta m})' - (\cos\phi_0 + \beta \sin\phi_0 + \cdots\cdots)(r_0' \ell_{\xi} m)$$
$$= r_0' (\beta \sin\phi_0 - \frac{\beta^2}{2}\cos\phi_0) \qquad (2.24)$$

2.3 Resultant Stresses and Couples:

The resultant stresses and couples are defined in the following manner:

$$N_\xi = \int_{-h/2}^{h/2} \sigma_\xi \, d\zeta \,, \qquad N_\theta = \int_{-h/2}^{h/2} \sigma_\theta \cdot d\zeta \,, \qquad Q = \int_{-h/2}^{h/2} \tau_{\xi\zeta} \, d\zeta$$

$$M_\theta = \int_{-h/2}^{h/2} \sigma_\theta \, \zeta \cdot d\zeta \,, \qquad M_\xi = \int_{-h/2}^{h/2} \sigma_\xi \, \zeta \, d\zeta$$

(2.25)

Units for resultant stresses and couples have been taken as pounds per inch and pound-inch per inch respectively.

Vector representation of resultant stresses and couples:

The resultant stresses and couples as defined by Equation (2.25) may now be combined to form resultant stress and couple vectors as follows:

$$\bar{N}_\xi = N_\xi \bar{J}_\xi + Q \bar{n} \,, \qquad \bar{N}_\theta = N_\theta \bar{J}_\theta$$
$$\bar{M}_\xi = M_\xi \bar{J}_\theta \,, \qquad \bar{M}_\theta = - M_\theta \bar{J}_\xi \qquad (2.26)$$

In addition to these vectors a load intensity vector $\bar{P}$ will be introduced in the form

$$\bar{P} = P_\xi \bar{J}_\xi + P_n \bar{n} \qquad (2.27)$$

For what follows it is convenient to write $\bar{N}_\xi$ and $\bar{P}$ in the alternate form

$$\bar{N}_\xi = H\bar{J}_r + V\bar{k}, \qquad \bar{P} = P_H\bar{J}_r + P_V\bar{k} \tag{2.28}$$

where the radial stress resultant H and axial stress resultant V are related to N_ξ and Q as follows:

$$N_\xi = H\cos\phi + V\sin\phi, \qquad Q = -H\sin\phi + V\cos\phi \tag{2.29}$$

and the relation between load intensity P and its axial and radial components will be written as

$$P_H = P\sin\phi, \qquad P_V = -P\cos\phi \tag{2.30}$$

2.4 Differential Equations of Equilibrium:

The stress resultants N_ξ, N_θ, and Q, and the stress couples, M_θ and M_ξ, along with load intensity P acting on an element of shell are shown in Figure (6). The resulting force equilibrium condition for elements of shell may now be written in the following vectorial representation:

$$\frac{\partial}{\partial\xi}(r\bar{N}_\xi) + \frac{\partial}{\partial\theta}(\alpha\bar{N}_\theta) + r\alpha\bar{P} = 0 \tag{2.31}$$

Similarly, the moment equilibrium condition for elements of shell is

$$\frac{\partial}{\partial\xi}(r\bar{M}_\xi) + \frac{\partial}{\partial\theta}(\alpha\bar{M}_\theta) + r\alpha(\bar{J}_\xi \times \bar{N}_\xi + \bar{J}_\theta \times \bar{N}_\theta) = 0 \tag{2.32}$$

Now, substituting for $\bar{N}_\theta$, $\bar{N}_\xi$, $\bar{M}_\theta$, $\bar{M}_\xi$ and $\bar{P}$ from Equations (2.26) to (2.30) into above two equations of equilibrium, we get the following three scalar equilibrium conditions for elements of shell (see Appendix 3):

$$r\alpha P_V + (rV)' = 0$$

$$(rH)' - \alpha N_\theta + r\alpha P_H = 0$$

$$(rM_\xi)' - r' M_\theta - r\alpha Q = 0 \qquad (2.33\ a,b,c)$$

2.5 Stress-Strain Relations:

Assuming that the behaviour of the material is isotropic, the stress resultants and couples are related to direct and bending strains by the equations (see Appendix 4)

$$C e_{\xi m} = N_\xi - \nu N_\theta \,, \qquad C e_{\xi m} = N_\theta - \nu N_\xi \qquad (2.34\ a,b)$$

$$M_\xi = D(\kappa_\xi + \nu \kappa_\theta) \,, \qquad M_\theta = D(\kappa_\theta + \nu \kappa_\xi) \qquad (2.35\ a,b)$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)} \,, \qquad C = Eh \qquad (2.36\ a,b)$$

E is the modulus of elasticity and ν is poisson's ratio.

Summary

The following equations represent seven equations for seven quantities u, β, H, V, N_θ, M_ξ and M_θ, and thus form a complete system of equations for the problem.

i) $(rV)' + r\alpha P_V = 0$

ii) $(rH)' - \alpha N_\theta + r\alpha P_H = 0$

iii) $(rM_\xi)' - r' M_\theta - r\alpha Q = 0$ (2.33 a,b,c)

iv) $C e_{\xi m} = N_\xi - \nu N_\theta$

v) $C e_{\theta m} = N_\theta - \nu N_\xi$ (2.34 a,b)

vi) $M_\xi = D(\kappa_\xi + \nu \kappa_\theta)$

vii) $M_\theta = D(\kappa_\theta + \nu \kappa_\xi)$ (2.35 a,b)

PART 3

SMALL-DEFLECTION THEORY OF THIN SHELLS OF REVOLUTION

The small deflection theory follows from the foregoing analysis by referring the differential equations of equilibrium to the undeformed shell and also by omitting all the non-linear terms in the expressions for strain as derived in Part 2.

3.1 System of Equations for Small-deflection Theory:

For small displacements the equations for strain, after omitting all the non-linear terms in the expressions for strain as derived in Part 2 and dropping subscript o, take the following form:

$$e_{\xi m} = \frac{u' - z'\beta}{r'}, \qquad e_{\theta m} = \frac{u}{r} \qquad (3.1\ a;b)$$

$$\chi_\theta = \frac{r'\beta}{r\alpha}, \qquad \chi_\xi = \frac{\beta'}{\alpha} \qquad (3.2\ a;b)$$

Equations (3.1) immediately imply the compatibility relation

$$(re_{\theta m})' - re_{\xi m} = z'\beta \qquad (3.3)$$

The differential equations of equilibrium as referred to the undeformed shell are as before

$$(rV)' + r\alpha P_V = 0$$

$$(rH)' - \alpha N_\theta + r\alpha P_H = 0$$

$$(rM_\xi)' - r'M_\theta - r\alpha Q = 0 \qquad (3.4\ a;b,c)$$

The stress-strain equations as given in Part 2 do not change and are rewritten

$$C e_{\xi m} = N_\xi - \nu N_\theta , \qquad C e_{\theta m} = N_\theta - \nu N_\xi \qquad (3.5\text{a;b})$$

$$M_\xi = D(\chi_\xi + \nu \chi_\theta), \qquad M_\theta = D(\chi_\theta + \nu \chi_\xi) \qquad (3.6\text{a;b})$$

3.2 Reduction to Two Simultaneous Equations:

The next and perhaps the most important step in the theory is to reduce the system of Equations (3.1) to (3.6) to two simultaneous differential equations of second order for the two variables β and (rH). By combining Equations (3.2), (3.6) and (3.4c) we get (see Appendix 5)

$$\beta'' + \frac{(rD/\alpha)'}{(rD/\alpha)}\beta' - \left[\left(\frac{r'}{r}\right)^2 - \nu\frac{(r'D/\alpha)'}{(rD/\alpha)}\right]\beta + \frac{z'}{(rD/\alpha)}(rH) = \frac{r'}{(rD/\alpha)}(rV)$$

and by combining Equations (3.3), (3.4b), and (3.5) we get the second differential equation of the following form:

$$\begin{aligned}(rH)'' + \frac{(r/C\alpha)'}{(r/C\alpha)}(rH)' - \left[\left(\frac{r'}{r}\right)^2 + \nu\frac{(r'/C\alpha)'}{(r/C\alpha)}\right](rH) - \frac{z'}{(r/C\alpha)}\beta \\ = \left[\frac{z'r'}{r^2} + \nu\frac{(z'/C\alpha)'}{(r/C\alpha)}\right](rV) + \nu\frac{z'}{r}(rV)' \\ - \left[\frac{(r/C\alpha)'}{(r/C\alpha)} + \nu\frac{r'}{r}\right](r\alpha P_H) - (r\alpha P_H)'\end{aligned}$$

$$(3.7\text{a;b})$$

For a particular loading, (rV) is given by (3.4a). Equations (3.7) are then solved for β and (rH). The stress resultants are expressed in terms of these solutions and the components of load intensity by Equations (3.4b) and

(2.29). The stress couples are determined by Equations (3.2) and (3.6). The displacement u results from either of Equations (3.1) in conjunction with Equations (3.5). The displacement ω is obtained from the equation

$$\omega = \int (z' e_{\xi m} - r'\beta)\, d\xi \tag{3.8}$$

PART 4

ANALYSIS OF TOROIDAL SHELL OF ELLIPTICAL CROSS-SECTION SUBJECTED TO UNIFORM NORMAL PRESSURE

For a toroidal shell of elliptical cross-section, the parametric equation (2.1) may be taken in the form

$$r = a + b\sin\xi, \qquad z = -c\cos\xi \tag{4.1}$$

where b and c are the semiaxes of the cross-section and a is the distance of the centre line, see Figure (7). According to Equations (2.2) we then have

$$\alpha = \sqrt{b^2\cos^2\xi + c^2\sin^2\xi} = c\sqrt{1 - e^2\cos^2\xi} \tag{4.2}$$

where

$$e^2 = 1 - \left(\frac{b}{c}\right)^2 \tag{4.3}$$

It will be convenient to define the parameters

$$\lambda = \frac{b}{a}, \qquad \mu = \gamma\left(\frac{bc}{ah}\right), \qquad \gamma = \sqrt{12(1-\nu^2)} \tag{4.4}$$

Next, let p denote a uniform internal pressure intensity. Radial and axial components of load intensity are given by

$$\alpha p_H = z' p, \qquad \alpha p_V = -r' p \tag{4.5}$$

If we now introduce the dimensionless stress function

$$\Psi = (\gamma/Eh^2)(rH) \tag{4.6}$$

and assume that the thickness h of shell is constant, using Equations (4.1) to (4.6) we find that the two simultaneous differential equations (3.7a,b) assume the form,

(see Appendix 6)

$$\beta'' + \left(\frac{\lambda \cos\xi}{1+\lambda \sin\xi} - \frac{e^2 \sin\xi \cos\xi}{1-e^2\cos^2\xi}\right)\beta' - \left[\left(\frac{\lambda\cos\xi}{1+\lambda\sin\xi}\right)^2 + \frac{\nu\lambda\sin\xi}{(1+\lambda\sin\xi)(1-e^2\cos^2\xi)}\right]\beta$$

$$+ \frac{\mu\sin\xi}{1+\lambda\sin\xi}\sqrt{\frac{1-e^2\cos^2\xi}{1-e^2}}\,\Psi = \frac{\mu\cos\xi}{1+\lambda\sin\xi}\sqrt{1-e^2\cos^2\xi}\;\gamma(rV)/Eh^2.$$

$$\Psi'' + \left(\frac{\lambda\cos\xi}{1+\lambda\sin\xi} - \frac{e^2\sin\xi\cos\xi}{1-e^2\cos^2\xi}\right)\Psi' - \left[\frac{\lambda\cos\xi}{1+\lambda\sin\xi} - \frac{\nu\lambda\sin\xi}{(1+\lambda\sin\xi)(1-e^2\cos^2\xi)}\right]\Psi$$

$$- \frac{\mu\sin\xi}{1+\lambda\sin\xi}\sqrt{\frac{1-e^2\cos^2\xi}{1-e^2}}\,\beta = \left[\frac{c}{b}\,\frac{\lambda^2\sin\xi\cos\xi}{(1+\lambda\sin\xi)^2} + \frac{b}{c}\,\frac{\nu\lambda\cos\xi}{(1+\lambda\sin\xi)(1-e^2\cos^2\xi)}\right]\frac{\gamma(rV)}{Eh^2}$$

$$- \left[\frac{(1-e^2)\cos\xi}{(1-e^2\cos^2\xi)}(1+\lambda\sin\xi) + 2\lambda\sin\xi\cos\xi\right]\gamma a c p/Eh^2. \qquad (4.7\;a;b)$$

On the basis of the assumption that the ratio $\lambda = \frac{b}{a}$ is small compared to unity, we can neglect all terms in our Equations (4.7) which involve λ. The basic differential equations of the problem assume now the following form,

$$\beta'' - \frac{e^2 \sin\xi \cos\xi}{1 - e^2\cos^2\xi}\beta' + \mu \sin\xi \sqrt{\frac{1 - e^2\cos^2\xi}{1 - e^2}}\,\Psi = \mu \cos\xi \sqrt{1 - e^2\cos^2\xi}\ \frac{\gamma(rV)}{Eh^2}$$

$$\Psi'' - \frac{e^2 \sin\xi \cos\xi}{1 - e^2\cos^2\xi}\Psi' - \mu \sin\xi \sqrt{\frac{1 - e^2\cos^2\xi}{1 - e^2}}\,\beta$$

$$= -\left[\frac{(1 - e^2)\cos\xi}{1 - e^2\cos^2\xi}\right]\frac{\gamma a c p}{Eh^2} \qquad (4.8\text{ a; b})$$

The next step is to eliminate the first derivative term in each of Equations (4.8). This is arrived at by introducing two new variables X and Y defined by relations of the following form (see Appendix 7)

$$\beta = X(1 - e^2\cos^2\xi)^{1/4}, \qquad \Psi = Y(1 - e^2\cos^2\xi)^{1/4} \qquad (4.9\text{ a; b})$$

and if we substitute Equations (4.9) into (4.8), the resulting two differential equations may be written as

$$X'' + X\left[\frac{1}{2}\frac{e^2\cos 2\xi}{1 - e^2\cos^2\xi} - \frac{5}{16}\left(\frac{e^2\sin 2\xi}{1 - e^2\cos^2\xi}\right)^2\right]$$

$$+\mu \sin\xi \sqrt{\frac{1 - e^2\cos^2\xi}{1 - e^2}}\ Y = \mu\cos\xi\,(1 - e^2\cos^2\xi)^{1/4}\,\gamma(rV)/Eh^2$$

$$Y'' + Y\ \frac{1}{2}\left[\frac{e^2\cos 2\xi}{1 - e^2\cos^2\xi} - \frac{5}{16}\left(\frac{e^2\sin 2\xi}{1 - e^2\cos^2\xi}\right)^2\right] - \mu\sin\xi\sqrt{\frac{1 - e^2\cos^2\xi}{1 - e^2}}\ X$$

$$= -\left[\frac{(1 - e^2)\cos\xi}{(1 - e^2\cos^2\xi)^{3/4}}\right]\frac{\gamma a c p}{Eh^2}. \qquad (4.10\text{ a; b})$$

The above differential equations may now be written in the simpler form

$$X'' + \theta X + \mu\Phi Y = F$$

$$Y'' + \theta Y - \mu\Phi X = G \qquad (4.11\text{ a; b})$$

where the functions Θ, Φ, F and G are given as

$$\Theta = \left[\frac{1}{2}\frac{e^2\cos 2\xi}{1-e^2\cos^2\xi} - \frac{5}{16}\left(\frac{e^2\sin 2\xi}{1-e^2\cos^2\xi}\right)^2\right] \tag{4.12}$$

$$\Phi = \sin\xi\sqrt{\frac{1-e^2\cos^2\xi}{1-e^2}} \tag{4.13}$$

$$F = \mu\cos\xi(1-e^2\cos^2\xi)^{1/4}\frac{\gamma}{Eh^2}(rV) \tag{4.14}$$

$$G = -\left[\frac{(1-e^2)\cos\xi}{(1-e^2\cos^2\xi)^{5/4}}\right]\frac{\gamma acp}{Eh^2} \tag{4.15}$$

Asymptotic solution of differential equations:

Approximate solutions of differential equations (4.11a,b) can often be found by methods of asymptotic integration provided the parameter μ in these equations is sufficiently large compared to unity. The form and completeness of the results depend on the nature of coefficient functions Θ and Φ. For instance, if these functions are bounded from zero, asymptotic expansions in μ can be found which represent solutions of either the non-homogeneous differential equations or the corresponding homogeneous equations and which involve only elementary functions. Such expansions have been given for general shells of revolution by F. B. Hildebrand (3). If Φ vanishes at some transition point, asymptotic solutions can be obtained for the homogeneous differential equation using methods developed by R. E. Langer (4) which involve Bessel functions. In general, Langer's methods yield only the leading terms of asymptotic expansions of solutions but may be used to find higher-order terms if the shell is such that Equations (4.11a,b) can be reduced to a single second order differential equation.

The present solution is concerned with asymptotic solution of non-homogeneous equation where the coefficient functions Θ and Φ are bounded functions of parameter ξ for the shell and that coefficient Φ vanishes at some isolated point. According to Equation (4.13), Φ vanishes at $\xi = 0$, and π or, in general, whenever the tangent to a meridian is perpendicular to the axis of revolution.

The function Θ as defined by Equation (4.12) is of order unity, (see Appendix 8) and so is small compared to $\mu\Phi$ where μ has been taken sufficiently large. This suggests that approximate solutions of differential equations (4.11a,b) can be found by neglecting Θ. The resulting differential equation may now be interpreted as the real and imaginary parts of a single complex differential equation of the form

$$Z'' - i\mu\Phi Z = f \tag{4.16}$$

where

$$Z = X + iY, \quad f = F + iG \tag{4.17}$$

If we assume that a solution can be represented by an expansion in inverse powers of μ, we obtain by substituting

$$Z = \left(A_0 + \frac{A_1}{\mu} + \frac{A_2}{\mu^2} + \frac{A_3}{\mu^3} + \cdots\cdots\right) \tag{4.18}$$

into (4.16) an approximate solution, (see Appendix 9)

$$Z = \frac{if}{\mu\Phi} + \frac{1}{\mu^2\Phi}\left(\frac{f}{\Phi}\right)'' + \cdots\cdots \tag{4.19}$$

which obviously is not valid in the neighborhood of $\xi = 0$ and π where the function Φ vanishes. However, over an interval where Φ is bounded from zero, a finite number of terms of (4.19) does furnish an approximate solution of Equation (4.16).

The basic idea now to be applied is to obtain a formula for an approximate solution of Equation (4.16) which satisfies Equation (4.16) approximately, for small values of ξ and behaves like Equation (4.19) when ξ is not small.

R. A. Clark (2) has shown that in the immediate neighborhood of $\xi = 0$ differential equation (4.16) may be approximated by

$$Z'' - i\mu\xi Z = f(0) \tag{4.20}$$

where we assume that $f(0) \neq 0$. A solution of (4.20) is given by

$$Z = \mu^{-2/3} f(0)\, T(\mu^{1/3}\xi) \tag{4.21}$$

provided the function T(x) is a solution of

$$T''(x) - ixT(x) = 1 \tag{4.22}$$

Let $T(x)$ stand for that (unique) solution of (4.22) which has the behaviour $T(x) \sim i/x$ as $|x| \to \infty$ and consider the expression

$$Z_1(\xi) = \mu^{-2/3} f(\xi)\, T(\mu^{1/3}\Phi), \tag{4.23}$$

near $\xi = 0$, $\Phi \sim \xi$ and $f(\xi) \sim f(0)$ so that Z_1 approximates a solution of (4.20). Away from $\xi = 0$, $\mu^{1/3}\Phi$ is large for large μ so that $Z_1 \sim i f/\mu\Phi$. Consequently Z_1 furnishes an interpolation between a solution of Equation (4.20) and the approximate solution of (4.16) as given by (4.19). The real and imaginary parts of the required function $T(x)$ and its derivative are tabulated in a paper by R. A. Clark (1).

APPENDICES

APPENDIX 1

Derivation of Equation (2.6)

$$\bar{R} = r\bar{J}_r + z\bar{k} + \zeta\bar{n}$$

$$d\bar{R} = \frac{\partial \bar{R}}{\partial \xi} d\xi + \frac{\partial \bar{R}}{\partial \theta} d\theta + \frac{\partial \bar{R}}{\partial \zeta} d\zeta$$

$$\frac{\partial \bar{R}}{\partial \xi} = r'\bar{J}_r + z'\bar{k} + \zeta \frac{\partial \bar{n}}{\partial \xi} = r'\bar{J}_r + z'\bar{k} + \zeta \frac{\partial}{\partial \xi}(-\bar{J}_r \sin\phi + \bar{k}\cos\phi)$$

$$= \bar{J}_r(r' - \zeta\phi'\cos\phi) + \bar{k}(z' - \zeta\phi'\sin\phi)$$

$$\frac{\partial \bar{R}}{\partial \theta} = r\bar{J}_\theta + \zeta\frac{\partial \bar{n}}{\partial \theta} = r\bar{J}_\theta + \zeta\frac{\partial}{\partial \theta}(-\bar{J}_r \sin\theta + \bar{k}\cos\theta)$$

$$= \bar{J}_\theta(r - \zeta\sin\phi)$$

$$\frac{\partial \bar{R}}{\partial \zeta} = \bar{n}$$

Therefore $d\bar{R} = \left[\bar{J}_r(r - \zeta\phi'\cos\phi) + \bar{k}(z' - \zeta\phi'\sin\phi)\right]d\xi + \bar{J}_\theta(r - \zeta\sin\phi)d\theta + \bar{n}d\zeta$

Putting value of $\bar{n}$, we get

$$d\bar{R}\cdot d\bar{R} = d\xi^2(r'^2 - 2r\zeta\phi'\cos\phi + \zeta^2\phi'^2\cos^2\phi + z'^2 - 2z'\zeta\sin\phi + \zeta^2\phi'^2\sin^2\phi)$$
$$+ d\zeta^2(\cos^2\phi + \sin^2\phi) + r^2\left(1 - \zeta\frac{\sin\phi}{r}\right)^2 d\theta^2 + 2d\zeta d\xi(z'\cos\phi - r'\sin\phi)$$
$$= d\xi^2(\alpha^2 + \zeta^2\phi'^2 - 2\zeta\phi'\alpha) + d\zeta^2 + r^2\left(1 - \zeta\frac{\sin\phi}{r}\right)^2 d\theta^2 + 2d\zeta d\xi$$
$$= (\alpha - \zeta\phi')^2 d\xi^2 + r^2\left(1 - \zeta\frac{\sin\phi}{r}\right)^2 d\theta^2 + d\zeta^2$$

Now,

$$ds^2 = d\bar{R}\cdot d\bar{R}$$
$$= (\alpha - \zeta\phi')^2 d\xi^2 + r^2\left(1 - \zeta\frac{\sin\phi}{r}\right)^2 d\theta^2 + d\zeta^2 \qquad (2.6)$$

APPENDIX 2

Development of ℓ_{gm} and X_Θ in powers of β :

Maclaurin expansion of f(x)

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots\cdots$$

Development of ℓ_{gm} :

$$f(\beta) = \frac{1}{\cos(\phi_o - \beta)} \cos\phi_o\left(1 + \frac{u'}{r_o'}\right) - 1$$

$$f(0) = \frac{u'}{r_o'}, \quad f'(\beta) = -\cos\phi_o\left(1 + \frac{u'}{r_o'}\right)\frac{\tan(\phi_o - \beta)}{\cos(\phi_o - \beta)}$$

$$f'(0) = -\left(\tan\phi_o + \frac{u'}{r_o'}\beta\tan\phi_o\right)$$

Thus we can write,

$$\ell_{gm} = \left(\frac{u'}{r_o'}\right) - \beta\tan\phi_o + \beta^2\left(\frac{1}{2} + \tan^2\phi_o\right) - \left(\frac{u'\beta}{r_o'}\right)\tan\phi_o + \cdots\cdots \tag{2.19}$$

Development of X_Θ:

$$f(\beta) = -\frac{\sin(\phi_o - \beta) + \sin\phi_o}{r_o},$$

$$f(0) = 0, \quad f'(\beta) = \frac{\cos(\phi_o - \beta)}{r_o}, \quad f'(0) = \frac{\cos\phi_o}{r_o}$$

$$f''(\beta) = \frac{\sin(\phi_o - \beta)}{r_o},$$

$$f''(0) = \frac{\sin\phi_o}{r_o}$$

$$X_\Theta = \frac{\beta\cos\phi_o}{r_o} + \frac{\beta^2\sin\phi_o}{2r_o} + \cdots\cdots \tag{2.18}$$

Development of ω':

$$\omega' = \alpha \sin\phi - z_0'$$
$$= \alpha_0(1 + e_{\xi m}) \sin\phi - z_0'$$
$$= \alpha_0(1 + e_{\xi m}) \sin(\phi - \beta) - z_0' \qquad (2.20)$$

$$f(0) = z_0 e_{\xi m}$$
$$f'(\beta) = -\alpha_0(1 + e_{\xi m}) \cos(\phi_0 - \beta)$$
$$f''(\beta) = -\alpha_0(1 + e_{\xi m}) \sin(\phi_0 - \beta)$$
$$f'(0) = -r_0'(1 + e_{\xi m}), \quad f''(0) = -z_0(1 + e_{\xi m})$$

$$\omega' = z_0' e_{\xi m} - r_0'(1 + e_{\xi m})\beta - z_0'(1 + e_{\xi m})\beta^2/2 + \cdots\cdots \qquad (2.22)$$

Compatibility relation:

$$e_{\xi m} = \frac{\cos\phi_0}{\cos\phi}\left(1 + \frac{u'}{r_0'}\right) - 1,$$
$$e_{\theta m} = \frac{u}{r_0}.$$

Now, $u' = (r_0 e_{\theta m})'$, $\quad e_{\xi m} = \frac{\cos\phi_0}{\cos\phi}\left[1 + \frac{(r_0 e_{\theta m})'}{r_0'}\right] - 1$

$$r_0' \cos\phi \, e_{\xi m} = \cos\phi_0 \left[r_0' + (r_0 e_{\theta m})'\right] - r_0' \cos\phi$$

rearranging,

$$\cos\phi_0 (r_0 e_{\theta m})' - \cos\phi (r_0' e_{\xi m}) = r_0'(\cos\phi - \cos\phi_0) \qquad (2.23)$$

APPENDIX 3

Derivation of scalar equilibrium conditions from two vector equilibrium equations:

$$\bar{N}_\theta = \bar{J}_\theta N_\theta\,,\quad \bar{M}_\xi = \bar{J}_\theta M_\xi\,,\quad \bar{M}_\theta = -M_\theta \bar{J}_\xi\,,\quad \bar{N}_\xi = N_\xi \bar{J}_\xi + Q\bar{n} \qquad (2.26)$$

$$\bar{N}_\xi = \bar{J}_r H + \bar{k} V\,,\qquad \bar{P} = P_H \bar{J}_r + P_V \bar{k} \qquad (2.28)$$

$$\text{Also,}\quad N_\xi = H\cos\phi + V\sin\phi\,,\quad Q = -H\sin\phi + V\cos\phi\,. \qquad (2.29)$$

$$\frac{\partial}{\partial\xi}(r\bar{N}_\xi) + \frac{\partial}{\partial\theta}(\alpha\bar{N}_\theta) + r\alpha\bar{P} = 0$$

$$\frac{\partial}{\partial\xi}(rH\bar{J}_r + rV\bar{k}) + \frac{\partial}{\partial\theta}(\alpha N_\theta \bar{J}_\theta) + r\alpha(P_H\bar{J}_r + P_V\bar{k}) = 0$$

$$(rH)'\bar{J}_r + (rV)'\bar{k} + (\alpha N_\theta)(-\bar{J}_r) + r\alpha P_H\bar{J}_r + r\alpha P_V\bar{k} = 0$$

Thus, in radial direction $\quad \bar{J}_r\left[(rH)' - \alpha N_\theta + r\alpha P_H\right] = 0$

and in axial direction $\quad \bar{k}\left[(rV)' + r\alpha P_V\right] = 0$

From vector moment equilibrium equation:

$$\frac{\partial}{\partial\xi}(r\bar{M}_\xi) + \frac{\partial}{\partial\theta}(\alpha\bar{M}_\theta) + r\alpha(\bar{J}_\theta \times \bar{N}_\theta + \bar{J}_\xi \times \bar{N}_\xi) = 0$$

$$\frac{\partial}{\partial\xi}(rM_\xi j_\theta) + \frac{\partial}{\partial\theta}(-\alpha M_\theta j_\xi) + r\alpha\left[\bar{J}_\xi \times (Hj_r + Vk) + \underbrace{j_\theta \times j_\theta}_{=0} N_\theta\right]$$

$$\left[(rM_\xi)' - \alpha M_\theta \cos\phi\right]\bar{j}_\theta + r\alpha(H\sin\phi - V\cos\phi)\bar{j}_\theta = 0$$

$$(rM_\xi)' - r'M_\theta - r\alpha Q = 0$$

Thus we have three scalar equations:

$$(rH)' - \alpha N_\theta + r\alpha P_H = 0$$

$$(rM_\xi)' - r'M_\theta - r\alpha Q = 0$$

$$(rV)' + r\alpha P_V = 0 \qquad (2.33\ a, b, c)$$

APPENDIX 4

Derivation of stress-strain relations:

As the effect of transverse shear and transverse normal stress on the deformation is neglected, we have the usual relations for an isotropic medium,

$$(1-\nu^2)\sigma_\xi = E(\sigma_\xi + \nu e_\theta)$$

$$(1-\nu^2)\sigma_\theta = E(\sigma_\theta + \nu e_\xi)$$

Now from (2.17)

$$e_\xi = e_{\xi m} + \zeta\chi_\xi, \qquad e_\theta = e_{\theta m} + \zeta\chi_\theta$$

$$N_\xi = \int_{-h/2}^{h/2} \sigma_\xi \, d\zeta$$

$$= \frac{E}{(1-\nu^2)} \int_{-h/2}^{h/2} \left[e_{\xi m} + \zeta\chi_\xi + \nu(e_{\theta m} + \zeta\chi_\theta)\right] d\zeta$$

$$= \frac{E}{(1-\nu^2)} \left[e_{\xi m}\zeta + \zeta^2/2 + \nu e_{\theta m}\zeta + \nu\zeta^2\chi_\theta\right]_{-h/2}^{h/2}$$

$$= \frac{E}{(1-\nu^2)} \left[h(e_{\xi m} + \nu e_{\theta m}) + h^2/4\,(\chi_\xi + \nu\chi_\theta)\right]$$

Similarly,

$$N_\theta = \frac{E}{(1-\nu^2)} \left[h(e_{\theta m} + \nu e_{\xi m}) + h^2/4\,(\chi_\theta + \nu\chi_\xi)\right]$$

$$N_\xi - \nu N_\theta = \frac{E}{(1-\nu^2)} \left[h(1-\nu^2)(e_{\xi m} + h/4)\right]$$

Assuming thin shell, h/4 being very small, neglecting h/4

$$N_\xi - \nu N_\theta = \frac{E}{(1-\nu^2)}\left[h(1-\nu^2)\, e_{\xi m}\right] = E h e_{\xi m}$$

Thus $\quad C e_{\xi m} = N_\xi - \nu N_\theta$ (2.34a)

where C = Eh

Similarly it can be shown that

$$C e_{\theta m} = N_\theta - \nu N_\xi \tag{2.34b}$$

$$M_\xi = \int_{-h/2}^{h/2} \sigma_\xi \cdot \varsigma \cdot d\varsigma$$

$$= \frac{E}{1-\nu^2}\int_{-h/2}^{h/2}\left[e_{\xi m} + \varsigma \chi_\xi + \nu e_{\theta m} + \varsigma \nu \chi_\theta\right] d\varsigma$$

$$= \frac{2E}{1-\nu^2}\left[e_{\xi m}\, \varsigma^2/2 + \varsigma^3/3\, \chi_\xi + \varsigma^2/2\, \nu e_{\theta m} + \nu\, \varsigma^3/3\, \chi_\theta\right]_{-h/2}^{h/2}$$

$$= \frac{E}{1-\nu^2}\left[\frac{h^3}{12}(\chi_\xi + \nu \chi_\theta) + \frac{h^2}{4}(e_{\xi m} + \nu e_{\theta m})\right]$$

$$= \frac{E h^3}{12(1-\nu^2)}\left(\chi_\xi + \nu \chi_\theta\right),$$

$h^2/4$ term being small.

$$M_\xi = D(\chi_\xi + \nu \chi_\theta),$$

where $D = \dfrac{E h^3}{12(1-\nu^2)}$

(2.35a)

Similarly, $\quad M_\theta = D(\chi_\theta + \nu \chi_\xi)$ (2.35b)

APPENDIX 5

Derivation of Equations (3.7a,b):

$$(rM_\xi)' - r'M_\theta - r\alpha Q = 0 \tag{3.4c}$$

Now substituting values for M_ξ, M_θ, and Q we get,

$$\left[\left(\frac{rD}{\alpha}\right)\left(\beta' + \nu\frac{r'}{r}\right)\right]' - \frac{r'D}{\alpha}\left(\frac{r'}{r}\beta + \nu\beta'\right) + (rH)z' - (rV)r' = 0$$

$$\left(\frac{rD}{\alpha}\right)'\beta' + \left(\frac{rD}{\alpha}\right)\beta'' + \nu D\left(\frac{r'}{\alpha}\right)'\beta + \nu\left(\frac{r'D}{\alpha}\right)\beta' - \frac{D}{\alpha}\frac{(r')^2}{r}\beta - \nu\frac{r'D}{\alpha}\beta'$$

$$+ (rH)z' - (rV)r' = 0$$

Rearranging after simplification we get,

$$\beta'' + \frac{(rD/\alpha)'}{(rD/\alpha)}\beta' - \left[\left(\frac{r'}{r}\right)^2 - \nu\left(\frac{r'D/\alpha}{rD/\alpha}\right)'\right]\beta + \frac{z'}{(rD/\alpha)}(rH)$$
$$= \frac{r'}{(rD/\alpha)}(rV) \tag{3.7a}$$

Substitute values for $e_{\xi m}$ and $e_{\theta m}$ into compatibility equations. We get

$$\left[\frac{r}{C}(N_\xi - \nu N_\theta)\right]' - \frac{r'}{C}(N_\theta - \nu N_\xi) = z'\beta$$

$$\left[(rH)\frac{r'}{C\alpha} + (rV)\frac{z'}{C\alpha} - \nu\frac{r}{C\alpha}(rH)' - \nu\frac{r}{C}(rP_H)\right]'$$
$$- \left[(rH)\frac{r'}{C\alpha} + (rP_H)\frac{r'}{C} - \nu(rH)\frac{r'^2}{Cr\alpha} - (rV)\nu\frac{z'r'}{Cr\alpha}\right]$$
$$= z'\beta$$

On differentiating the left hand side term we get,

$$
\begin{aligned}
(rH)''\left(-\nu\frac{r}{C\alpha}\right) + (rH)'\left(\frac{\nu r}{C\alpha}\right) + (rH)\left[\left(\frac{r'}{C\alpha}\right)' + \nu\,\frac{r'^2}{Cr\alpha}\right] \\
= z'\beta - (rV)'\left(\frac{z'}{C\alpha}\right) - (rV)\left[\left(\frac{z'}{C\alpha}\right)' + \nu\,\frac{z'r'}{Cr\alpha}\right] + (r\,P_H)'\left(\frac{\nu r}{C}\right) \\
+ (r\,P_H)\left[\frac{r'}{C} + \nu\left(\frac{r}{C}\right)'\right].
\end{aligned}
$$

On re-arranging, the above expression reduces to equation (3.7, b).

APPENDIX 6

Derivation of Equations (4.7a,b):

From Equations (3.7a,b)

$$\frac{rD}{\alpha} = \frac{D}{c}(a+b\sin\xi)(1-e^2\cos^2\xi)^{-1/2}$$

$$\left(\frac{rD}{\alpha}\right)' = \frac{D}{c}\left[(1-e^2\cos^2\xi)^{-1/2}(b\cos\xi) - (a+b\sin\xi)(1-e^2\cos^2\xi)\,e^2\cos^2\xi\sin\xi\right]$$

$$\left(\frac{r'}{r}\right)^2 = \frac{\lambda^2\cos^2\xi}{(1+\lambda\sin\xi)^2}$$

$$\frac{r'D}{\alpha} = \frac{D}{c}(b\cos\xi)(1-e^2\cos^2\xi)^{-1/2}$$

$$\left(\frac{r'D}{\alpha}\right)' = \frac{D}{c}(b\cos\xi)(1-e^2\cos^2\xi)^{-1/2}$$

$$\left(\frac{r'D}{\alpha}\right)' = \frac{D}{c}\left[-b\sin\xi(1-e^2\cos^2\xi)^{-1/2} - (b\cos\xi)\,e^2\cos\xi\sin\xi\,(1-e^2\cos^2\xi)^{-3/2}\right]$$

$$\frac{(r'D/\alpha)'}{(rD/\alpha)} = -\left[\frac{\lambda\sin\xi}{(1+\lambda\sin\xi)(1-e^2\cos^2\xi)}\right]$$

$$\frac{z'}{rD/\alpha} = \frac{c^2\sin\xi\sqrt{1-e^2\cos^2\xi}}{aD(1+\lambda\sin\xi)} = \frac{bc}{ah}\gamma\frac{\sin\xi}{(1+\lambda\sin\xi)}\sqrt{\frac{(1-e^2\cos^2\xi)}{1-e^2}}\frac{\gamma}{Eh^2}$$

$$\frac{z'}{rD/\alpha}(rH) = \mu\frac{\sin\xi}{1+\lambda\sin\xi}\sqrt{\frac{1-e^2\cos^2\xi}{1-e^2}}$$

$$\frac{r'}{rD/\alpha}(rV) = \mu\cos\xi\sqrt{(1-e^2\cos^2\xi)}\;\gamma\frac{(rV)}{Eh^2}$$

$$\frac{z'}{r/C\alpha} = Eh\frac{cb}{a}\sqrt{\frac{1-e^2\cos^2\xi}{1-e^2}}$$

$$= \mu\sqrt{\frac{1-e^2\cos^2\xi}{1-e^2}}\cdot\frac{\sin\xi}{1+\lambda\sin\xi}\cdot\frac{Eh^2}{\gamma}$$

$$\frac{\dot z \dot r}{r^2} = \frac{c}{b}\,\frac{\lambda^2 \sin\xi \cos\xi}{(1+\lambda\sin\xi)^2}$$

$$\frac{z'}{\alpha} = \sin\xi\,(1-e^2\cos^2\xi)^{-1/2}$$

$$\left(\frac{z'}{\alpha}\right)' = \left[\frac{\cos\xi}{(1-e^2\cos^2\xi)^{1/2}} - \frac{e^2\cos\xi\sin\xi}{(1-e^2\cos^2\xi)^{3/2}}\right] = \frac{b^2}{c^2}\,\frac{\cos\xi}{(1-e^2\cos^2\xi)^{3/2}}$$

$$\frac{r}{\alpha} = \frac{b}{c}\,\frac{1+\lambda\sin\xi}{\lambda(1-e^2\cos^2\xi)}$$

$$\frac{(z/\alpha)'}{(r/\alpha)} = \frac{b}{c}\,\frac{\lambda\cos\xi}{(1-e^2\cos^2\xi)(1+\lambda\sin\xi)}$$

$$\nu\frac{z'}{r'}(rV)' = \nu\frac{z'}{r}(-r\alpha p_V) = \nu\, acp\cdot\lambda\cos\xi\sin\xi$$

$$(r\alpha p_H) = acp\,\sin\xi\,(1+\lambda\sin\xi)$$

$$(r\alpha p_H)' = acp(\cos\xi + 2\lambda\cos\xi\sin\xi)$$

$$\nu\frac{z'}{r}(rV)' - \left[\frac{(r/C\alpha)'}{(r/C\alpha)} + \nu\frac{\dot r}{r}\right](r\alpha p_H) - (r\alpha p_H)'$$

$$= acp\nu\lambda\sin\xi\cos\xi - \left[\frac{\lambda\cos\xi}{1+\lambda\sin\xi} - \frac{e^2\cos\xi\sin\xi}{(1-e^2\cos^2\xi)} + \frac{\nu\lambda\cos\xi}{1+\lambda\sin\xi}\right]$$

$$\times\, acp\sin\xi - acp(\cos\xi + 2\lambda\sin\xi\cos\xi)$$

$$= acp\left[-\cos\xi(1+\lambda\sin\xi) + \frac{e^2\cos\xi\sin\xi(1+\lambda\sin\xi)}{(1-e^2\cos^2\xi)} - 2\lambda\cos\xi\sin\xi\right]$$

$$= -acp\left[\frac{(1-e^2)\cos\xi}{(1-e^2\cos^2\xi)}(1+\lambda\sin\xi) + 2\lambda\sin\xi\cos\xi\right]$$

Combining above terms we get equations (4·7, a,b)

APPENDIX 7

To eliminate first derivative terms in (4.8a,b)

$$\beta = X(1-e^2\cos^2\xi)^{1/4}, \quad \Psi = Y(1-e^2\cos^2\xi)^{1/4} \qquad (4.9\,a,b)$$

Substitute into (4.8a,b)

$$\beta' = X'(1-e^2\cos^2\xi)^{1/4} + \frac{X}{4}(1-e^2\cos^2\xi)^{-3/4} e^2 \sin 2\xi$$

$$\beta'' = X''(1-e^2\cos^2\xi)^{1/4} + \frac{X'}{2}(1-e^2\cos^2\xi)^{-3/4} e^2 \sin 2\xi$$

$$+ X\ \tfrac{1}{2}(1-e^2\cos^2\xi)^{3/4} e^2\cos 2\xi - \tfrac{3}{16}(1-e^2\cos^2\xi)^{-7/4}(e^2\sin 2\xi)^2$$

Equation (4.8a) now becomes

$$X''(1-e^2\cos^2\xi)^{1/4} + \tfrac{1}{2}X'(1-e^2\cos^2\xi)^{-3/4} e^2 \sin 2\xi + X\left[\frac{e^2\cos 2\xi}{2}(1-e^2\cos^2\xi)^{-3/4}\right.$$

$$\left. - \tfrac{3}{16}(1-e^2\cos^2\xi)^{-7/4}(e^2\sin 2\xi)^2\right] - \frac{e^2 \sin 2\xi}{2(1-e^2\cos^2\xi)}\left[(1-e^2\cos^2\xi)^{1/4}X'\right.$$

$$\left. + X/4\,(1-e^2\cos^2\xi)^{-3/4} e^2 \sin 2\xi\right] + \mu \sin\xi \sqrt{\frac{1-e^2\cos^2\xi}{1-e^2}}\,(1-e^2\cos^2\xi)^{1/4}\, Y$$

$$= \mu \cos\xi \sqrt{1-e^2\cos^2\xi}\ \frac{\gamma(rV)}{Eh^2}$$

Dividing both sides by $(1-e^2\cos^2\xi)^{1/4}$, we get

$$X'' + X\,\frac{1}{2}\left[\frac{e^2\cos 2\xi}{1-e^2\cos^2\xi} - \frac{5}{16}\,\frac{(e^2\sin 2\xi)}{(1-e^2\cos^2\xi)}\right] + \mu \sin\xi \sqrt{\frac{1-e^2\cos^2\xi}{1-e^2}}\ Y$$

$$= \mu \cos\xi (1-e^2\cos^2\xi)^{1/4}\, \frac{\gamma(rV)}{Eh^2} \qquad \text{——— (4.8a)}$$

By similar substitution we get equation (4.8b)

APPENDIX 8

To show θ is of order unity:

$$\theta = \left[\frac{1}{2}\frac{e^2\cos 2\xi}{1-e^2\cos^2\xi} - \frac{5}{16}\frac{(e^2\sin 2\xi)^2}{(1-e^2\cos^2\xi)^2}\right]$$

when $\xi = 0, \pi$

$$\theta = \frac{e^2}{2(1-e^2)} - \frac{5}{16}(0)$$

$$= \frac{1}{2}\frac{e^2}{(1-e^2)} = O(1) \qquad \text{for } e^2 < 1$$

for $\xi = \frac{\pi}{2}$

$$\theta = -\frac{1}{2}e^2 - \frac{5}{16}(0)$$

$$= O(1)$$

for $\xi = \frac{\pi}{4}$

$$\theta = 0 - 5/16\, e^4$$

$$= O(1)$$

$$\Phi = \sin\xi\sqrt{\frac{1-e^2\cos^2\xi}{1-e^2}}$$

Away from $\xi = 0$, Φ is also of order unity.

Thus θ is of order unity and small compared to $\mu\Phi$ where μ is a large parameter.

APPENDIX 9

Solution of Equation (4.16) by asymptotic integration:

$$Z'' - i\mu\Phi Z = f \qquad (4.16)$$

Let $$Z = A_0 + \frac{A_1}{\mu} + \frac{A_2}{\mu^2} + \cdots\cdots \qquad (4.18)$$

Substituting in (4.16)

$$\frac{2A_1}{\mu^2} + \frac{6A_2}{\mu^3} + \cdots\cdots = f + i\mu\Phi\left(A_0 + \frac{A_1}{\mu} + \frac{A_2}{\mu^2} + \cdots\right)$$

Equating coefficients,

$$A_0 = A_2 = 0, \quad A_1 = if/\Phi, \quad A_4 = 2f/\Phi^2$$

Thus, $$Z = \frac{if}{\mu\Phi} + \frac{2f}{\mu^4\Phi^2} + \cdots\cdots$$

Now $$f = \frac{i}{\mu}\left(\frac{f}{\Phi}\right)'' + \frac{2}{\mu^4}\left(\frac{f}{\Phi^2}\right)'' - i\mu\Phi\left(\frac{if}{\mu\Phi} + \frac{2f}{\mu^4\Phi^2}\right)$$

On simplification, we get

$$\frac{2f}{\mu^2\Phi} = \left(\frac{f}{\Phi}\right)'' - i\frac{2}{\mu^3}\left(\frac{f}{\Phi^2}\right)''$$

which on further approximation becomes

$$\frac{2f}{\mu^2\Phi} = \left(\frac{f}{\Phi}\right)''.$$

Thus,

$$Z = \frac{if}{\mu\Phi} + \frac{1}{\mu^2\Phi}\left(\frac{f}{\Phi}\right)'' + \cdots\cdots$$

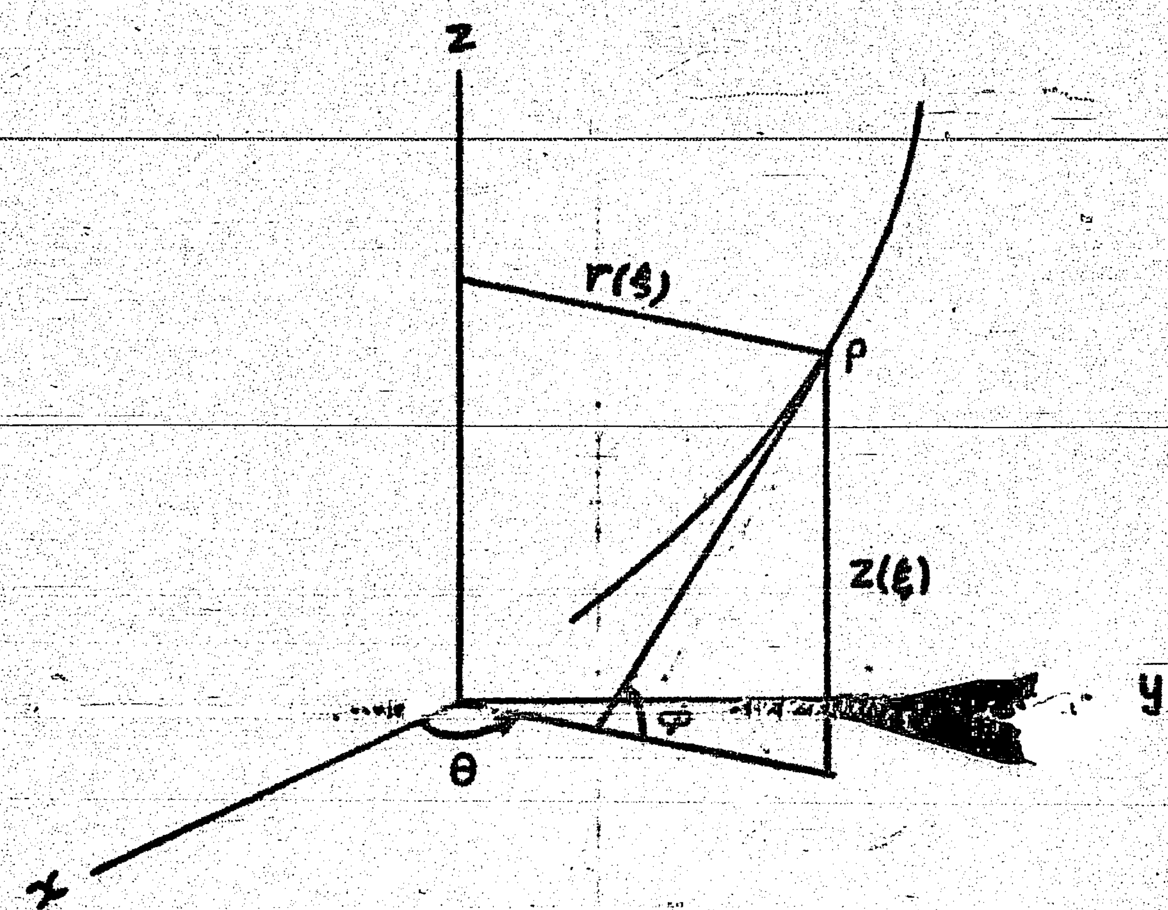

Figure 1 Coordinates

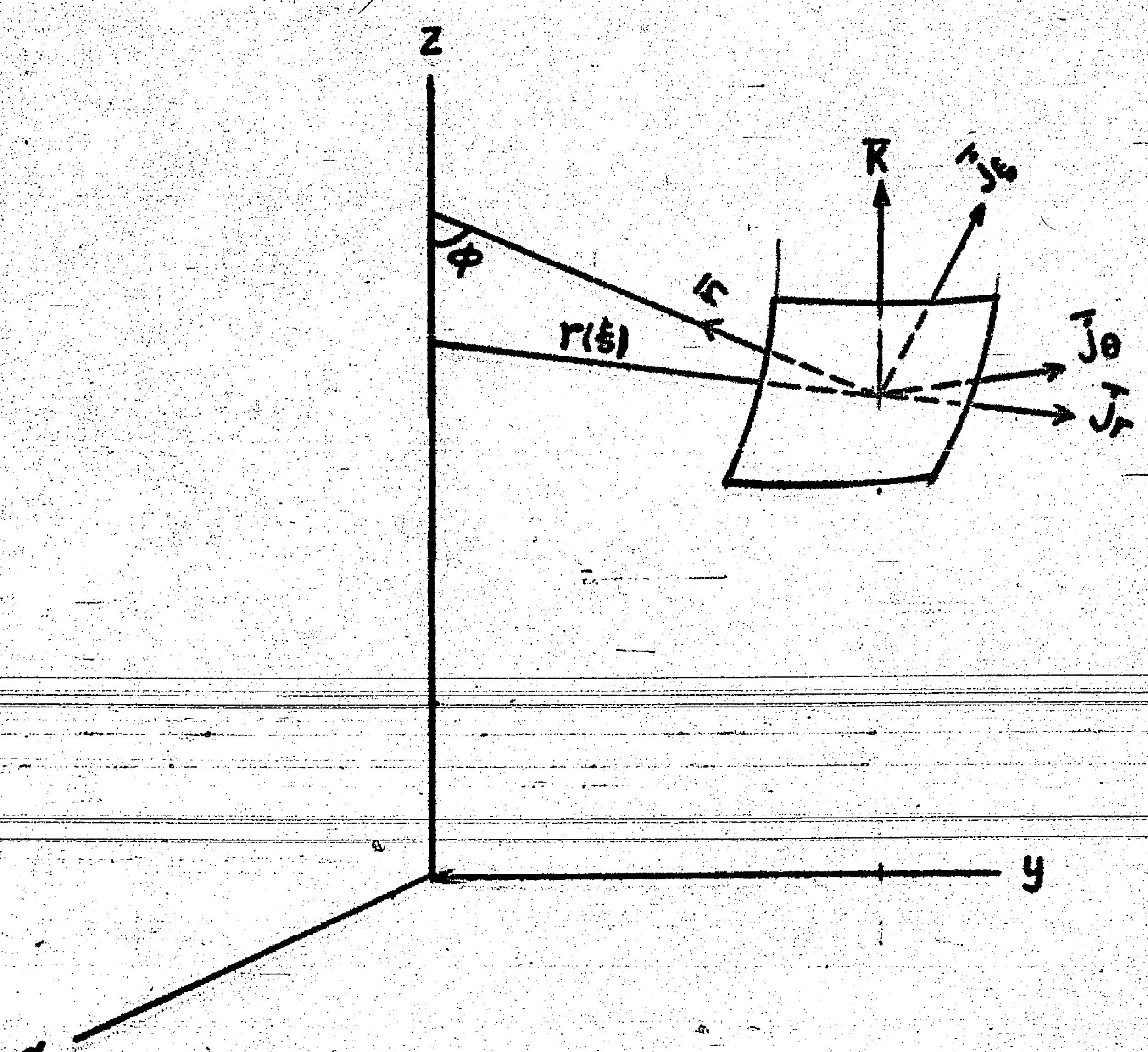

Figure 2 Unit Vectors

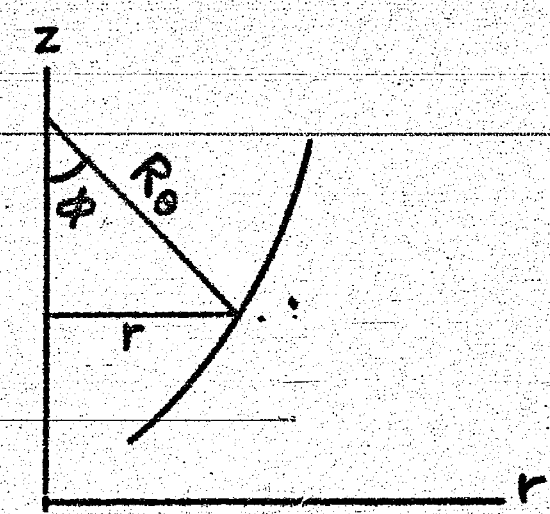

Figure 3

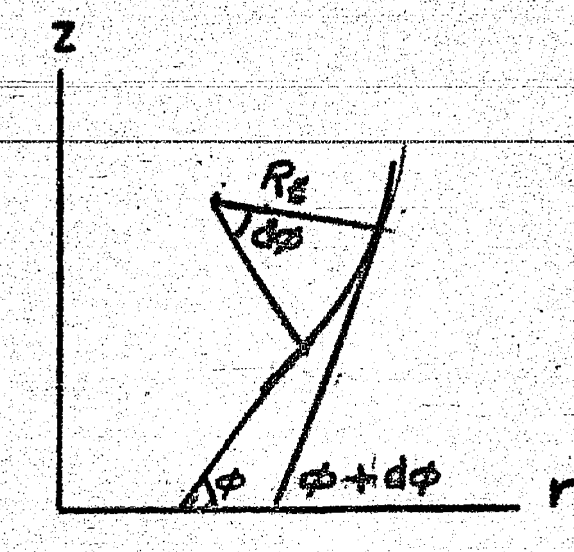

Figure 4

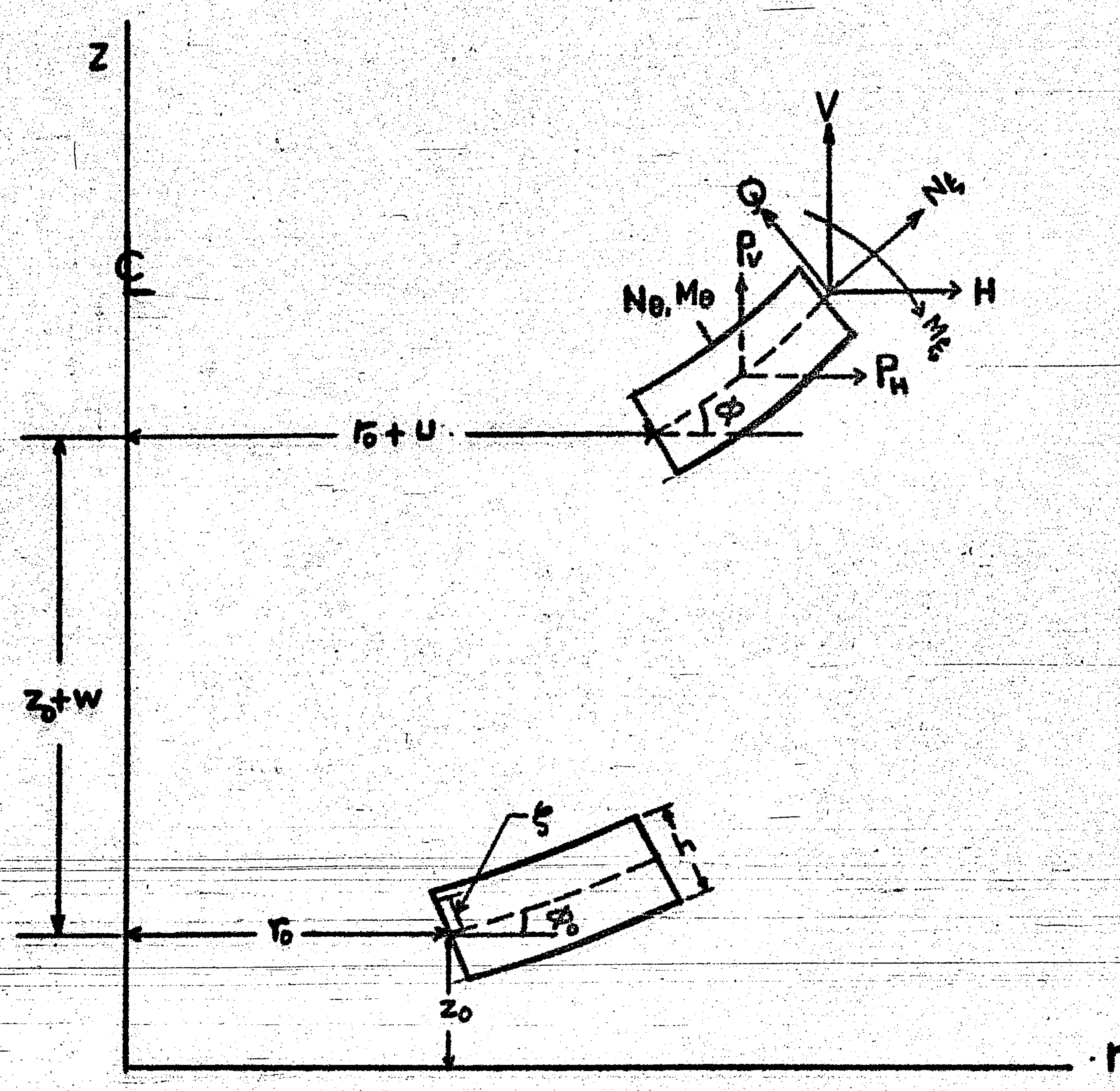

Figure 5 End View of Element for Deformed and Undeformed Positions

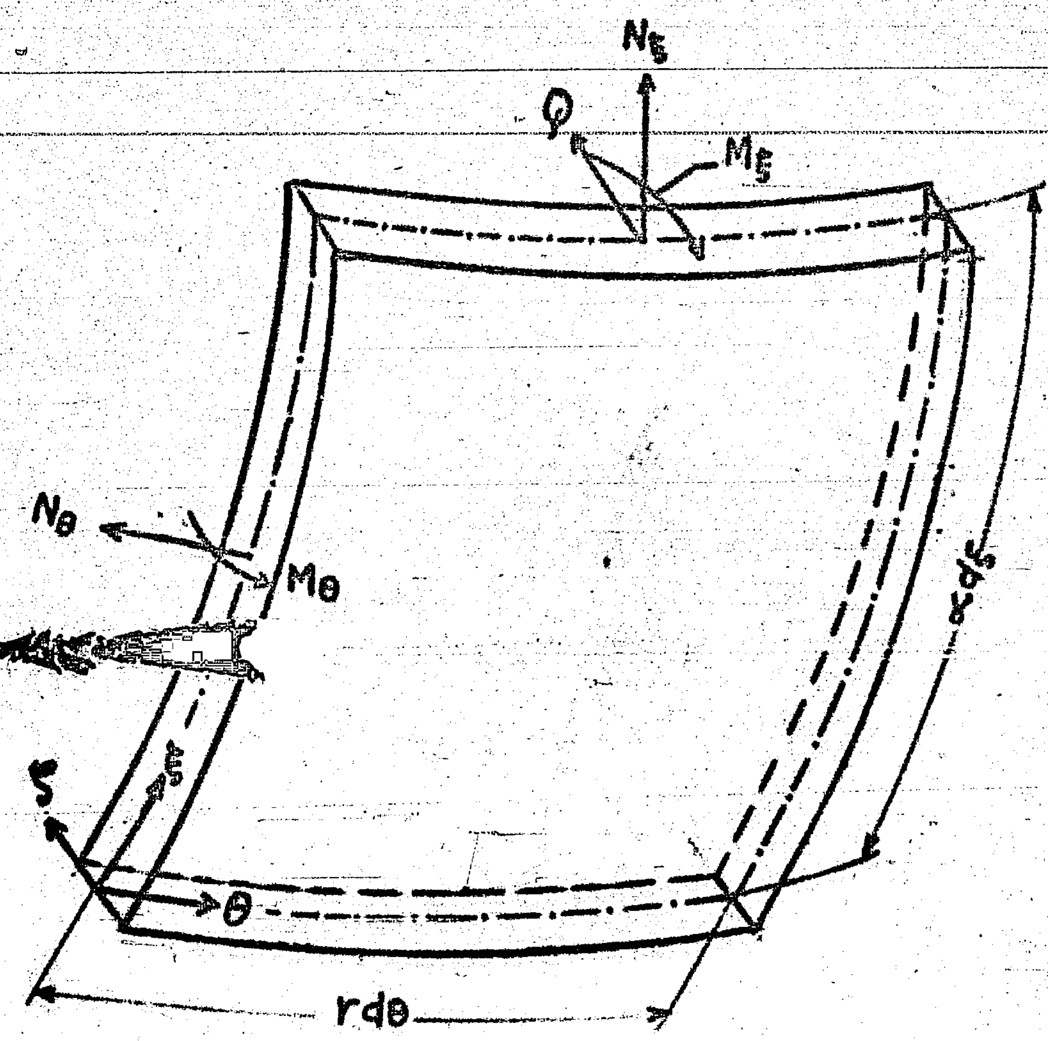

Figure 6 Resultant Stresses and Couples Shown for an Element of Shell

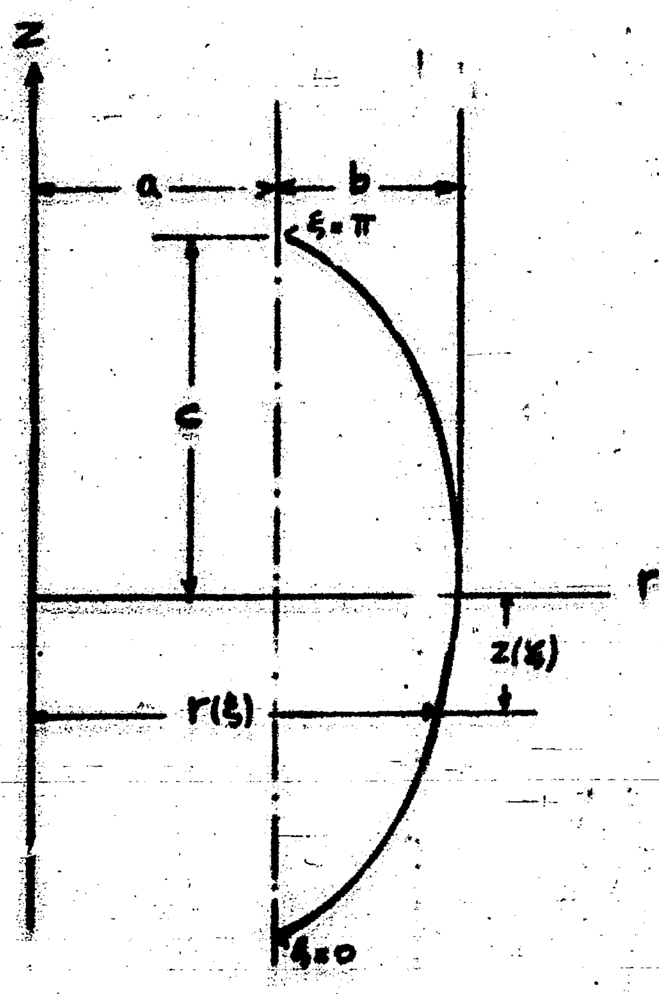

Figure 7 Elliptic Profile of the Shell

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